Markov Chains, SANs and Search Engines

Amy Langville

College of Charleston

May 3, 2002

Outline

- Markov chains and SANs
  - definitions
  - stationary analysis techniques
  - difficulties with MC and SAN analysis
  - NKP Preconditioner
- Search Engines
  - Introduction
  - VSM
  - Google and MCs

PART 1: MCs and SANs

Markov chains

- stochastic process which follows Markov property. $\Rightarrow$ State of system at time t only depends on the most recent past.

- DTMC

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n} \end{pmatrix},$$

where $p_{i,j}$ = conditional probability of moving from i to j in one time step.

- CTMC

$$Q = \begin{pmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,1} & q_{n,2} & \cdots & q_{n,n} \end{pmatrix},$$

where $q_{i,j}$ = transition rate of moving from i to j and $q_{i,i} = -\Sigma_{j \neq i} q_{i,j}$.

- Relationship between P and Q.

From MC Definition to MC Analysis

- Once we have P or Q for MC, we can begin analysis.

  There are two main types of analysis:

  - Transient analysis: find prob. dist. vector $\pi(t)$ at any time t
    - EX: $\pi_1(5)$ = prob. it will be good weather in Charleston 5 days from now.
  - Stationary analysis: find prob. dist. vector π as $t \to \infty$
    - EX: π_1 = long-run proportion of time Charleston will have good weather.

The Stationary Problem

- Now let $t \to \infty$.

  GOAL: Find π, the long-run steady-state probability vector.

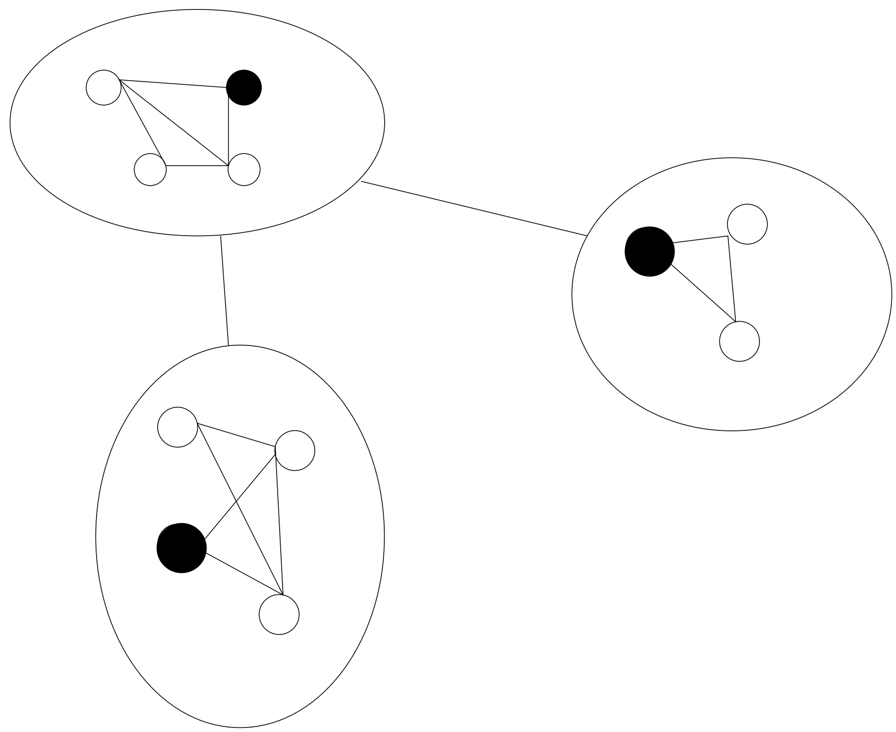
  π_i = proportion of time system spends in state i.

- The solution π is calculated from the homogeneous linear system

$$\pi Q = 0, \quad \pi e = 1.$$

- Or eigenvalue problem of finding e-vector associated with unit e-value.

$$\pi P = \pi, \quad \pi e = 1.$$



Stochastic Automata Networks (SANs)

- Problem with MC analysis = size of P or Q.
- One solution = SANs (represent Q in compact form)
- A SAN is a collection of stochastic automata which act more or less independently requiring only infrequent interaction.

Types of Infrequent Interaction

- *Functional Transitions*: rate at which transition may occur in one automaton may be function of states of other automata.
  - Example: Resource Sharing Model
- *Synchronizing events*: transition in one automaton may force transition to occur in one or more other automata.
  - Example: Two service centers in tandem

Kronecker Algebra: $\otimes, \oplus$ are key SAN operations.

- $A \in \Re^{m_1 \times n_1}$, $B \in \Re^{m_2 \times n_2}$, then $A \otimes B \in \Re^{m_1 m_2 \times n_1 n_2}$ and is defined by

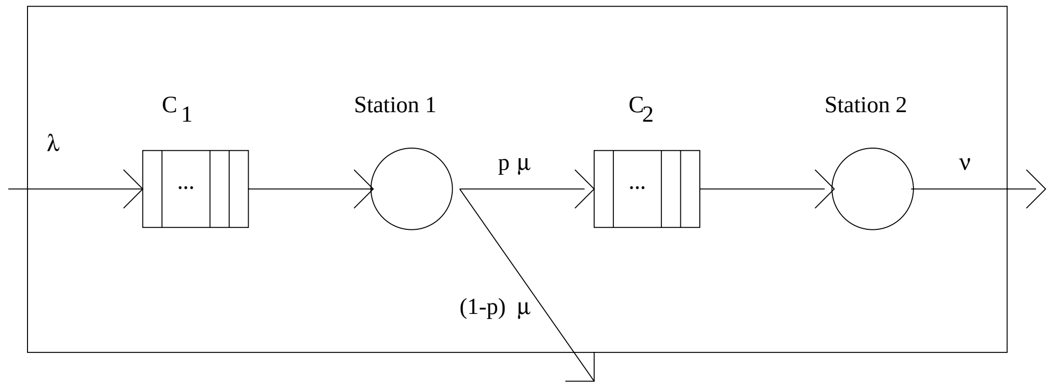
$$A \otimes B = \begin{pmatrix} a_{1,1}B & \cdots & a_{1,n_1}B \\ \vdots & \ddots & \vdots \\ a_{m_1,1}B & \cdots & a_{m_1,n_1}B \end{pmatrix}.$$

- $A \oplus B = A \otimes I_{m_2} + I_{m_1} \otimes B$. ($\oplus$ is defined in terms of $\otimes$.)

  (defined only for square matrices)

- Some Kronecker Properties:

  1. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
  2. $\oplus_{i=1}^{N} A^{(i)} = \Sigma_{i=1}^{N} I_{n_1} \otimes \cdots \otimes I_{n_{i-1}} \otimes A^{(i)} \otimes I_{n_{i+1}} \otimes \cdots \otimes I_{n_N}$.
  3. $(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$.
  4. $(A \otimes B)^{D} = A^{D} \otimes B^{D}$.



SAN Example: A Simple Queueing Network

Queueing network: two exponential, finite-capacity, single-server stations in tandem. When a station is full, customers are lost. For example, for $C_1 = 1$ and $C_2 = 2$, the transition rate matrix is given by

$$Q = \begin{pmatrix} -\lambda & 0 & 0 & \lambda & 0 & 0 \\ \nu & -(\nu+\lambda) & 0 & 0 & \lambda & 0 \\ 0 & \nu & -(\nu+\lambda) & 0 & 0 & \lambda \\ \mu(1-p) & \mu p & 0 & -\mu & 0 & 0 \\ 0 & \mu(1-p) & \mu p & \nu & -(\mu+\nu) & 0 \\ 0 & 0 & \mu(1-p) & 0 & \nu & -(\nu+\mu(1-p)) \end{pmatrix}.$$

Writing Q as a SAN

Use Kronecker notation to compress this.

- $A^{(1)}$ = station 1, $A^{(2)}$ = station 2.
- Separate local and synchronizing transitions.
  - Syn. Event 1 = departure from station 1 $\Rightarrow$ arrival to station 2
- Form $Q_l^{(1)}$ and $Q_l^{(2)}$.
- Form Q_{e_1}.

Writing Q as a SAN (cont.)

- $$Q_l^{(1)} = \begin{pmatrix} -\lambda & \lambda \\ \mu(1-p) & -\mu(1-p) \end{pmatrix}, \quad Q_l^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ \nu & -\nu & 0 \\ 0 & \nu & -\nu \end{pmatrix}.$$

- $$Q_{e_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mu p & 0 \\ 0 & 0 & \mu p \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -\mu p & 0 & 0 \\ 0 & -\mu p & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- We only need to store $Q_l^{(1)}$, $Q_l^{(2)}$, and the 4 matrices making up Q_{e_1}. For very large MCs, this is a huge savings in storage. The Global Q never needs to be formed or stored.

- Note that most of these small matrices are sparse.

Writing Q as a SAN (cont.)

- SAN represents the global generator matrix as

$$
\begin{aligned}
Q &= \oplus_{i=1}^{2} Q_l^{(i)} + \sum_{j=1}^{E} Q_{e_j} \\
&= Q_l^{(1)} \otimes I_{n_2} + I_{n_1} \otimes Q_l^{(2)} + \sum_{j=1}^{E} Q_{e_j}
\end{aligned}
$$

$$
= \begin{pmatrix} -\lambda & \lambda \\ \mu(1-p) & -\mu(1-p) \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ \nu & -\nu & 0 \\ 0 & \nu & -\nu \end{pmatrix}
$$

$$
+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mu p & 0 \\ 0 & 0 & \mu p \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -\mu p & 0 & 0 \\ 0 & -\mu p & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

SAN descriptor

- MC has global Q which can be stored in much more compact form as the ordinary sum of Kronecker products of much smaller matrices.

  $$Q = \sum_{j=1}^{2E+N} \otimes_{i=1}^{N} Q_j^{(i)},$$

  where

  $N =$ number of automata,

  $E =$ number of synchronizing events.

- Observations:
  - Effect of synchronizing events
  - Effect of functional transitions

SANs provide powerful storage savings, but at what cost?

Recall GOAL: fast stationary analysis

Stationary analysis of a SAN: $\pi\left(\Sigma_{j=1}^{2E+N} \otimes_{i=1}^{N} Q_j^{(i)}\right) = 0$.

- Review of methods employed
  - Direct: L, U factors hard to get
  - Iterative: Power–Stewart. Splittings–Dayar.
  - Projection: Arnoldi, GMRES–Stewart. BiCGSTAB, QMR–Buchholz.
- Problem: SANs $\downarrow$ storage, $\uparrow$ complexity $\Rightarrow$ time til convergence=high.
- Solutions:
  - Preconditioners - Stewart, Buchholz, Chan

Existing SAN Preconditioners

- ILU Preconditioners are hard to employ.
- Stewart's Neumann series inverse: M approximates $Q^{\#}$.

$$M = \Sigma_{h=0}^{H} P^h.$$

- Buchholz's preconditioner, similar to Stewart's.
- Additive/Multiplicative Schwartz
- Diagonal: $M = D^{-1}$, where D is the diagonal of Q.

Potential SAN Preconditioner: NKP ???

NKP Preconditioner for general matrix R

- Find A, B so that $R \approx A \otimes B$.
- Precondition with $M = A^{-1} \otimes B^{-1}$.

- Pitsianis and Van Loan:

$$\min \|R - A \otimes B\|_F^2 = \|\tilde{R} - a \circ b\|_F^2.$$

- requires $\tilde{R}$ and uses SVD.

The NKP, $A \otimes B$, when $R = \Sigma_{i=1}^{p} G_i \otimes F_i$

- It can be proven that when R has this special structure, the optimal A, B matrices are linear combinations of the matrices making up R.

$$\begin{aligned} A &= \alpha_1 G_1 + \alpha_2 G_2 + \cdots + \alpha_p G_p, \\ B &= \beta_1 F_1 + \beta_2 F_2 + \cdots + \beta_p F_p. \end{aligned}$$

- Original NKP problem can be written as NLP. Choose $\alpha_1, \alpha_2, \ldots, \alpha_p$, $\beta_1, \beta_2, \ldots, \beta_p$ so that this nonlinear function is minimized.

$$\|R - A \otimes B\|_F^2 = \| \sum_{i=1}^{p} G_i \otimes F_i - (\sum_{i=1}^{p} \alpha_i G_i) \otimes (\sum_{i=1}^{p} \beta_i F_i)\|_F^2.$$

Back to SANs

- $Q = \Sigma_{j=1}^{2E+N} \otimes_{i=1}^{N} Q_j^{(i)}$ has special structure similar to $R = \Sigma_{i=1}^{p} G_i \otimes F_i$.

  But Q is Kronecker product of N terms, not just 2 terms.

- SAN Problem: find $A, B, \ldots, N$ to min $\|Q - A \otimes B \otimes \cdots \otimes N\|_F^2$.

  Precondition with $M = A^{-1} \otimes B^{-1} \otimes \cdots \otimes N^{-1}$.

- Questions:

  - Finding A, B was based on the matrix SVD.

    Extension for finding $A, B, \ldots, N$?

  - When $R = \Sigma_{i=1}^{p} G_i \otimes F_i$, then $A = \Sigma_{i=1}^{p} \alpha_i G_i$, $B = \Sigma_{i=1}^{p} \beta_i F_i$.

    Extension to case when $R = Q = \Sigma_{j=1}^{2E+N} \otimes_{i=1}^{N} Q_j^{(i)}$?

Multilinear Algebra

- The tensor is fundamental object of multilinear algebra.
  - 1^{st}-order tensor is a vector.
  - 2^{nd}-order tensor is a matrix.
  - 3^{rd}-order tensor is a 3D box.
- Operations on Tensors:
  - inner product
  - outer product
  - scalar product
  - tensor-matrix multiplication
- Outer product of N vectors $(a \circ b \circ \cdots \circ n)$ results in an N^{th}-order tensor.

The HOSVD

- Tensors have rank as well as a SVD, called the HOSVD.
- A rank-1 N^{th}-order tensor is written as the outer product of N vectors.
- HOSVD of N^{th}-order tensor R shows that R can be written as the sum of rank-1 tensors.

$$R = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} s_{i_1,i_2,\cdots,i_N} u_{i_1}^{(1)} \circ u_{i_2}^{(2)} \circ \cdots \circ u_{i_N}^{(N)}.$$

Multilinear Algebra and the NKP

- We want to min $\|R - A \otimes B \otimes \cdots \otimes N\|_F^2$ for a general matrix R.

- Following the 2D case, we need to show $\exists \tilde{R} \ni$

$$\|R - A \otimes B \otimes \cdots \otimes N\|_F^2 = \|\tilde{R} - a \circ b \circ \cdots \circ n\|_F^2.$$

- $a \circ b \circ \cdots \circ n$ is a rank-1 N^{th}-order tensor

  $\Rightarrow$ rearrangement of R ($\tilde{R}$) must be N^{th}-order tensor.

- Goal: approximate N^{th}-order tensor with rank-1 N^{th}-order tensor

  $\Rightarrow$ use truncated HOSVD.

Question: How do we define this N^{th}-order tensor $\tilde{R}$?

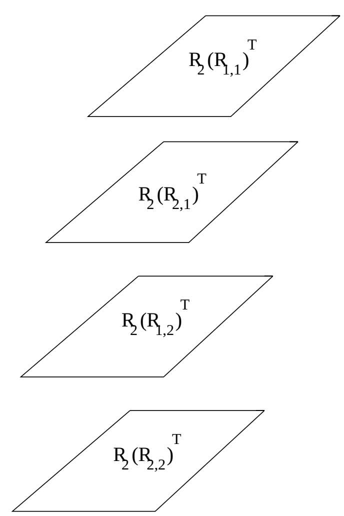
Defining Rearrangement Operator for Higher Orders

- For 2^{nd}-order tensor R, Pitsianis and Van Loan have already defined $\tilde{R} = R_2(R)$.
- The 3^{rd}-order rearrangement operator, $R_3(R)$, uses the 2^{nd}-order rearrangement operator.
- In general, we define the rearrangement operator recursively.

  We can then prove that with this recursive definition, the sums of squares match.

Example: Approximate $R_{12\times 12}$ by $A_{2\times 2}, B_{2\times 2}, C_{3\times 3}$.

$$\begin{aligned} R_{12\times 12} &= \begin{pmatrix} R_{1,1} & R_{1,2} \\ \\ R_{2,1} & R_{2,2} \end{pmatrix}. \\ \tilde{R}_{4\times 4\times 9} &= R_3(R) = \end{aligned}$$



and then

$$\|R - A \otimes B \otimes C\|_F^2 = \|R_3(R) - a \circ b \circ c\|_F^2.$$

Approximating N^{th}-order tensor by a rank-1 N^{th}-order tensor

GOAL: $\min \|R - A \otimes B \otimes \cdots \otimes N\|_F^2 = \min \|\tilde{R} - a \circ b \circ \cdots \circ n\|_F^2$, where $\tilde{R}$ is an N^{th}-order tensor and $a \circ b \circ \cdots \circ n$ is a rank-1 N^{th}-order tensor.

- Truncated HOSVD is not the optimal rank-1 approximation to the N^{th}-order tensor $\tilde{R}$.
- de Lathauwer:
  - truncated HOSVD is *good* approximation.
  - HO power algorithm to find *optimal* rank-1 approximation.
- We are only looking for an approximate inverse preconditioner.

  $\Rightarrow$ try truncating the HOSVD. Let

$$a = s_{1,1,\ldots,1} u_1^{(1)}, b = u_1^{(2)}, \ldots, n = u_1^{(N)}.$$

  We hope $A \otimes B \otimes \cdots \otimes N$ is a good approximation to R.

Solving $\| \Sigma_{j=1}^{T} \otimes_{i=1}^{N} Q_j^{(i)} - A \otimes B \otimes \cdots \otimes N \|_F^2$**, where** $T = 2E + N$**.**

- We can show

$$
\begin{aligned}
A &\approx \alpha_1 Q_1^{(1)} + \alpha_2 Q_2^{(1)} + \cdots + \alpha_T Q_T^{(1)} \\
B &\approx \beta_1 Q_1^{(2)} + \beta_2 Q_2^{(2)} + \cdots + \beta_T Q_T^{(2)} \\
&\vdots \\
N &\approx \eta_1 Q_1^{(N)} + \eta_2 Q_2^{(N)} + \cdots + \eta_T Q_T^{(N)}.
\end{aligned}
$$

- Thus, just as in the optimal 2-matrix case, the original problem

$$\min \|Q - A \otimes B \otimes \cdots \otimes N\|_F^2$$

can be transformed into NLP.

$$\|Q - A \otimes B \otimes \cdots \otimes N\|_F^2 \;\approx\; \| \sum_{j=1}^{T} \otimes_{i=1}^{N} Q_j^{(i)} - (\sum_{j=1}^{T} \alpha_j Q_j^{(1)}) \otimes (\sum_{j=1}^{T} \beta_j Q_j^{(2)}) \otimes \cdots \otimes (\sum_{j=1}^{T} \eta_j Q_j^{(N)})\|_F^2$$

$$= \left[\sum_{i=1}^{T} \sum_{j=1}^{T} \prod_{k=1}^{N} tr(Q_i^{(k)T} Q_j^{(k)})\right] - 2\left(\sum_{i=1}^{T}\left[\prod_{k=1}^{N}\left(\sum_{j=1}^{T} \alpha_j^{(k)} tr(Q_i^{(k)T} Q_j^{(k)})\right)\right]\right),$$

where $\alpha_j = \alpha_j^{(1)}, \beta_j = \alpha_j^{(2)}, \ldots, \eta_j = \alpha_j^{(N)}$.

With this transformation, $\tilde{Q}$ and its HOSVD are never needed.

- This is NLP of NT variables.

  As N and $E \uparrow$, practicality of problem transformation $\downarrow$.

  But, we can group automata!

Small artificial example with SAN structure

- $Q = \Sigma_{j=1}^{T} \otimes_{i=1}^{N} Q_j^{(i)}$, $N = 3$ and $T = 3$.

- So $Q = Q_1^{(1)} \otimes Q_1^{(2)} \otimes Q_1^{(3)} + Q_2^{(1)} \otimes Q_2^{(2)} \otimes Q_2^{(3)} + Q_3^{(1)} \otimes Q_3^{(2)} \otimes Q_3^{(3)}$.

- 9 matrices generated randomly.

- Generate NKP preconditioner $M = A^{-1} \otimes B^{-1} \otimes C^{-1}$ using NLP.

- Compare Q with $A \otimes B \otimes C$.

- Compare MQ with I.

(For rank $(n-1)$ SANs, compare with $QQ^{\#} = I - e\pi$.)

Small artificial example with SAN structure (cont.)

- Upper left block of Q:

$$\begin{pmatrix} 3.6900 & 2.5575 & 1.1395 & 0.7423 & 4.4001 & 2.8771 & 3.3112 \\ 0.6914 & 6.4420 & 0.5986 & 2.0307 & 0.9303 & 7.6925 & 0.5472 \\ 2.0044 & 1.4823 & 5.0960 & 3.5434 & 5.1132 & 3.5434 & 1.8152 \\ 0.4411 & 3.5066 & 0.3075 & 8.8268 & 0.4827 & 8.8753 & 0.4129 \\ 3.9318 & 2.6607 & 3.1883 & 2.1301 & 0.5820 & 0.4889 & 3.6444 \\ 0.2161 & 6.8078 & 0.6887 & 5.5756 & 0.4204 & 1.0499 & 0.1620 \\ 1.4760 & 1.3742 & 0.7873 & 0.6904 & 1.9557 & 1.3306 & 0.7284 \end{pmatrix}.$$

- Upper left block of $A \otimes B \otimes C$:

$$\begin{pmatrix} 3.6841 & 2.5885 & 1.0927 & 0.7678 & 4.3038 & 3.0240 & 3.3548 \\ 0.4798 & 6.4090 & 0.1423 & 1.9010 & 0.5605 & 7.4871 & 0.4369 \\ 2.0423 & 1.4350 & 5.1304 & 3.6048 & 5.1320 & 3.6059 & 1.8597 \\ 0.2660 & 3.5528 & 0.6681 & 8.9251 & 0.6683 & 8.9278 & 0.2422 \\ 3.9243 & 2.7573 & 3.1395 & 2.2059 & 0.6045 & 0.4247 & 3.5735 \\ 0.5111 & 6.8268 & 0.4089 & 5.4616 & 0.0787 & 1.0516 & 0.4654 \\ 1.4098 & 0.9906 & 0.4182 & 0.2938 & 1.6470 & 1.1572 & 0.6928 \end{pmatrix}.$$

Small artificial example with SAN structure (cont.)

- Upper left block of MQ:

$$\begin{pmatrix} 1.0920 & -0.0193 & -0.0947 & -0.0468 & -0.0727 & -0.0520 & 0.0201 \\ -0.0583 & 1.0465 & -0.0078 & -0.0976 & -0.0173 & -0.0828 & -0.0191 \\ 0.0284 & 0.0121 & 1.1466 & -0.0201 & 0.0649 & -0.0247 & 0.0163 \\ 0.0010 & 0.0283 & -0.0733 & 1.0891 & -0.0346 & 0.0389 & 0.0109 \\ -0.0331 & -0.0685 & 0.0113 & -0.0052 & 1.0676 & -0.0103 & -0.0337 \\ -0.0394 & -0.0597 & -0.0066 & 0.0064 & -0.0455 & 1.0317 & 0.0060 \\ -0.0171 & -0.0010 & -0.0126 & 0.0382 & -0.0166 & 0.0460 & 0.9939 \end{pmatrix}.$$

Thorough Testing on MCs and SANs

- MC Testing Motivation: Maybe NKP M will be low-storage, efficient preconditioner and compete with ILU M.

- SAN Testing Motivation: Find good SAN preconditioner.

Findings from MC Testing (cont.)

- Perturbation away from Kronecker structure:

Table 1: Effect of perturbations away from exact Kronecker product

$\|E\|_F$	$\|Q - NKP\|_F$	$\frac{\sigma_1}{\sum_{i=1}^{n} \sigma_i}$	status
.1726	.1571	.9999	succeeds
1.7263	1.4077	.9988	succeeds
5.1788	4.2848	.9965	succeeds
17.2627	14.8018	.9890	succeeds
51.9423	47.7126	.9676	succeeds
86.3134	74.8579	.9501	succeeds
127.2702	102.6748	.9346	fails
233.5594	201.3349	.8521	fails

- **Conclusion:** NKP M for MCs takes too much work and gives no benefit

  UNLESS there is some inherent Kronecker structure.

Encouraging for SAN NKP M

Testing NKP M on SANs

- Recall: Find $M = A^{-1} \otimes B^{-1} \otimes \cdots \otimes N^{-1}$ by NLP.

- Use M as preconditioner on SAN system $\pi(I - M(\Sigma_{j=1}^{2E+N} \otimes_{i=1}^{N} Q_j^{(i)})) = 0$.

- Outline for remainder of SAN talk:

  - Example 5B

  - Summary graphs: BiCGSTAB on example 5B

NKP M on SAN example 5B: Description

- Description: $N = 5$, $E = 4$. 5-server queueing network with respective capacities C_1, C_2, C_3, C_4, C_5.

- Size: $T = 13$, $NT = 65$.

- Diagram:

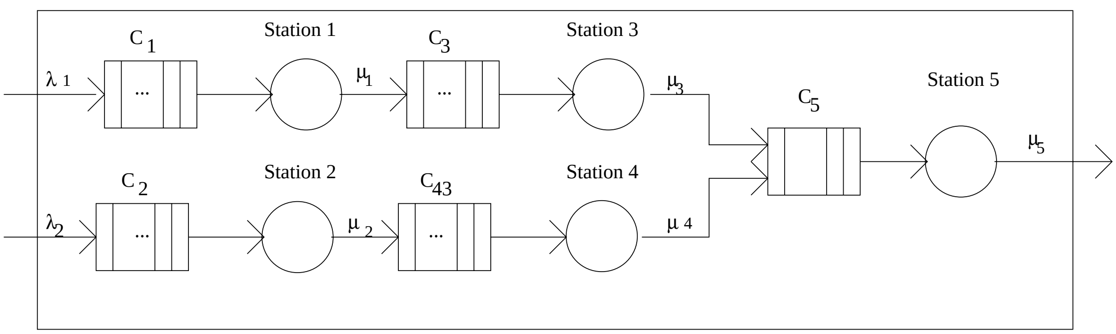


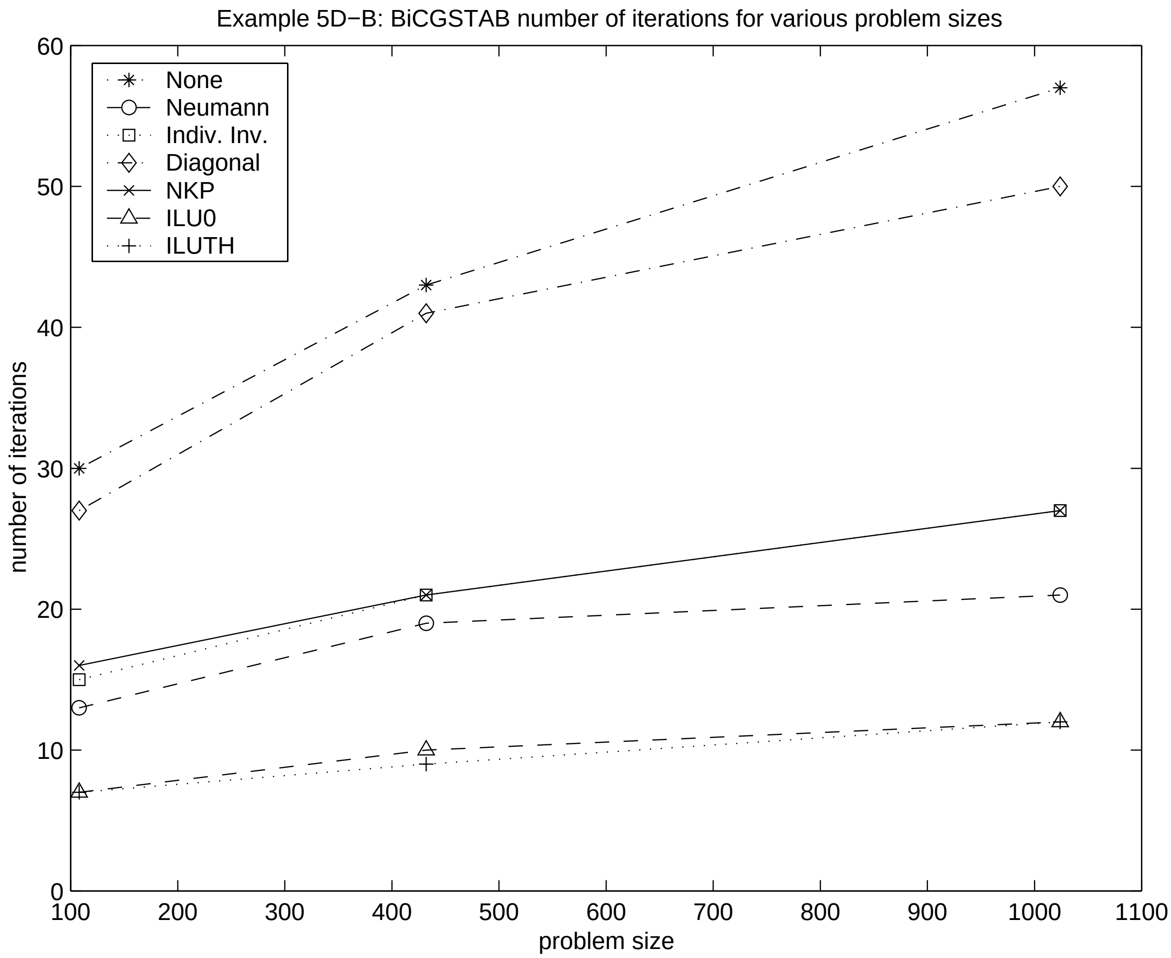
Figure 1: Example 5B: SAN queueing network with $N = 5$ automata

NKP M on SAN example 5B: Results

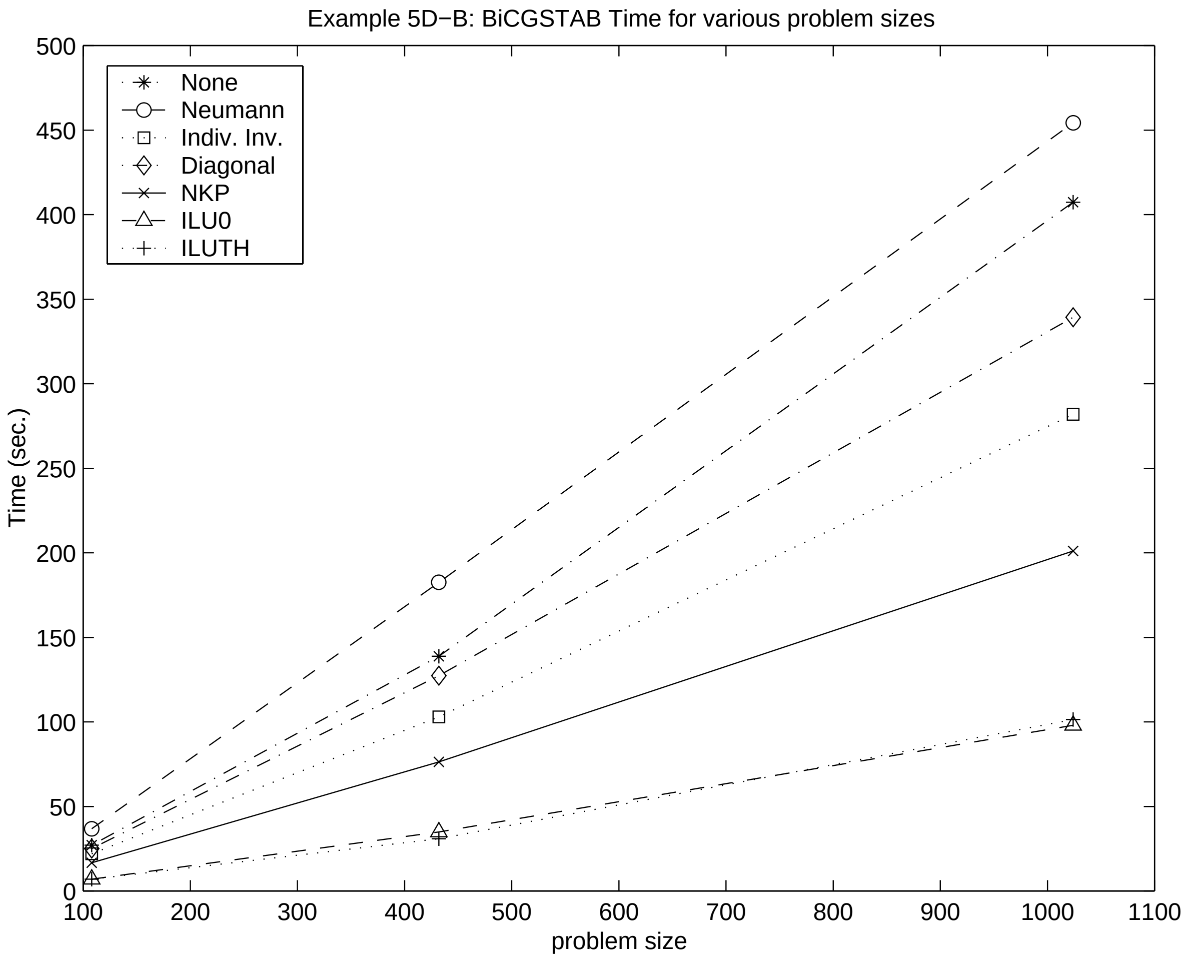
Table 2: Number of iterations and CPU times for stationary analysis of 5-D queueing network, example 5B

							Power		GMRES		BiCGSTAB	
C_1	C_2	C_3	C_4	C_5	order	Preconditioner	Iter.	Time	Iter.	Time	Iter.	Time
1	1	2	2	2	108	None	**144**	**66.38**	8	41.99	30	27.11
						Neumann	**48**	**67.15**	4	61.56	13	36.86
						Indiv. Inv.	**96**	**68.77**	5	40.26	15	22.00
						Diagonal	**261**	**123.34**	7	37.01	27	25.03
						NKP	**57**	**29.18**	4	23.26	16	16.69
						$ILU0$	**25**	**11.68**	3	15.80	7	6.95
						$ILUTH$	**19**	**8.82**	3	15.79	7	6.93
2	2	2	3	3	432	None	273	437.67	**11**	**203.08**	43	138.83
						Neumann	91	418.16	**5**	**270.05**	19	182.48
						Indiv. Inv.	156	378.43	**6**	**165.40**	21	102.92
						Diagonal	438	675.85	**10**	**178.18**	41	127.35
						NKP	103	174.25	**6**	**117.34**	21	76.37
						$ILU0$	40	65.00	**3**	**53.75**	10	34.97
						$ILUTH$	37	57.60	**3**	**56.68**	9	30.91
3	3	3	3	3	1024	None	301	988.21	12	487.73	**57**	**407.29**
						Neumann	100	983.18	6	676.42	**21**	**454.29**
						Indiv. Inv.	173	884.69	6	362.54	**27**	**281.89**
						Diagonal	375	1257.41	11	440.88	**50**	**339.30**
						NKP	123	463.85	7	309.91	**27**	**201.03**
						$ILU0$	49	168.73	4	169.12	**12**	**97.98**
						$ILUTH$	47	161.56	4	167.67	**12**	**101.39**

Summary Plots - BiCGSTAB (Example 5D-B)



Summary Plots - BiCGSTAB (Example 5D-B)



Verbal Summary of SAN NKP Tests

- NKP is best SAN preconditioner in terms of both storage and time.

  (ILU used for comparison purposes only.)

- Computation of NKP is worth effort.

  NKP computation ≤ 6 sec.

- Example 5D is representative example. $T = 13$, $NT = 65$ are typical limits after grouping is done. Size of each $A^{(i)}$ does not affect NLP.

PART 2: Search Engines

Outline

- Introduction
- Various IR models and measures
- VSM
- LSI
- MCs and IR
- WWW IR
- Related fields

Overview

Data Mining − gleaning facts, trends, observations from large collection of data

EX:
- minor/outlier clustering (stolen credit cards)
- building classes of customer profiles
- merging seemingly incompatible databases

Text Mining − mining document collections

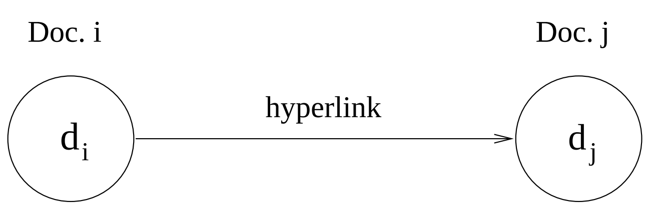
EX:
- determining authorities/hubs
- determining synonyms, semantic structure
- language translation

Information Retrieval

− retrieving documents in a collection most relevant to query

Some IR Models

- Boolean – EX: query = “Average Salary” **and** (US **or** Canada)
- Probabilistic
- Vector space models (VSM)
- Other:
  - Meta-search engines
  - Graph theoretic models – EX: WWW, node=doc, edge=hyperlink



IR Performance Measures

- Recall: maximize # useful docs

$$Recall = \frac{\#\ \ relevant\ \ docs\ \ retrieved}{\#\ \ relevant\ \ docs\ \ in\ \ collection}$$

- Precision: minimize # useless docs

$$Precision = \frac{\#\ \ relevant\ \ docs\ \ retrieved}{\#\ \ docs\ \ retrieved}$$

- Other:
  - Time
  - Storage
  - User frustration

LA + IR = VSM

- Term-by-Document Matrix $A_{m\times n}$

$$A_{m\times n} \quad = \quad \begin{pmatrix} 1 & 0 & 0 & .20 & .65 & .50 & .90 \\ 0 & 1 & .70 & .40 & .35 & .50 & .10 \\ 0 & 0 & .30 & .40 & 0 & 0 & 0 \end{pmatrix}$$

  - Variety of weighting schemes–Most common = tf.idf
  - Properties of $A_{m\times n}$:
    - A can be huge.
    - A is sparse but otherwise unstructured.
    - A may contain a lot of uncertainty (noise) due to synonymy, polysemy, indexer bias.

- Query vector q

$$q = \begin{pmatrix} 0 \\ \\ 0 \\ \\ 1 \end{pmatrix}.$$

- GOAL: determine which $d_1, d_2, \ldots, d_n$ are closest to q.

And do it FAST!

How to measure "closeness"?

- Length of vectors
- Similarity measures
- Angle measure between q and d_i for $i = 1 : n$.

$$cos(\theta_i) = \frac{q^T d_i}{\|q\| \|d_i\|}.$$

Let $\delta_i = cos(\theta_i)$. Retrieve doc. i if $\delta_i \geq tol$.

Problems with basic VSM

- size of A – can we compress A?

- noise in A – ambiguity in vocab. and writing style, polysems (bank), synonyms (car, automobile), indexing conventions (manual, automatic).

- Time for cosine calculations, $cos(\theta_i) = \frac{q^T d_i}{\|q\| \|d_i\|}$.

  – one solution: remove denom. calculation by normalizing

    so that $\|q\| = \|d_i\| = 1$.

  – But $q^T d_i$ for $i = 1 : n$ still costly, despite parallelization.

    $q^T d_i$ is inner product of m-dimensional vectors, where m is large,

    possibly $O(10^6)$.

- Need to: reduce "Noise" in A and do cosine calculation FAST!

Data Compression and Noise Reduction with LSI

- A has SVD, $A = U\Sigma V^T$.

- Use truncated SVD of A to capture semantic essence of A and drop noise.

$$A \approx A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T = U_k \Sigma_k V_k^T.$$

- Advantages:

  - A_k requires much less storage.

  - $cos(\theta_i)$ can be computed in projected space.

$$\begin{aligned} cos(\theta_i) &= \frac{q^T d_i}{\|q\|\|d_i\|} = \frac{q^T (Ae_i)}{\|q\|\|Ae_i\|} \\ &\approx \frac{q^T (A_k e_i)}{\|q\|\|A_k e_i\|} = \frac{q^T U_k \Sigma_k V_k^T e_i}{\|q\|\|U_k \Sigma_k V_k^T e_i\|} \\ &= \frac{q^T U_k s_i}{\|q\|\|s_i\|}, \end{aligned}$$

where $s_i = \Sigma_k V_k^T e_i$.

- s_i and $\|s_i\|$ can be computed once and stored for all i. After normalization, only 1 inner product between k-dimensional vectors is required. ($k << m$.)
- Latent semantic associations made.

- Disadvantages:
  - How to determine k? Empirical testing shows $k \leq 200$ works well.
  - Dynamic doc. collection – requires updating and downdating SVD.
  - Computing A_k, SVD, for large doc. collections is very costly, even if done only once a week at midnight.

Other Factorizations/Compression Techniques

- ULV^T
- SDD: $A_k = \sum_{i=1}^{k} \sigma_i x_i y_i^T$, where x_i, y_i use only {-1,0,1}.
- GSVD
- Riemann SVD
- ADE
- Concept Decomposition Cluster
- DFT–FFT?
- Wavelets?

MCs + IR = PageRank concept

- Google's PageRank concept:

$$PageRank(d_i) = \text{measure of importance of } d_i.$$

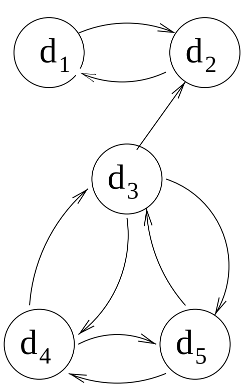
- Google uses Backlinks, we'll use MC π.

- Steps in Google search:

  1. enter query.

  2. direct search (full text scan) for group of docs containing query terms.

  3. sort by Google PageRank.



$$\text{GoogleA} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

- Replace with MCPageRank.

$$P = \begin{pmatrix} 0 & .9 & .1 & 0 & 0 \\ .9 & 0 & 0 & .1 & 0 \\ 0 & .2 & 0 & .7 & .1 \\ 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & .5 & .5 & 0 \end{pmatrix}.$$

- Google PageRank: involves dominant eigenvector computation.

- MC PageRank: involves finding π.

$$\pi_i = \text{proportion of time random user accesses } d_i.$$

- Google: distributes probability of moving from d_i to d_j equally along all outgoing links. MC is more flexible, could use web log usage files to fill in p_{ij}.

IR on the WEB

- Unique, Challenging Features:
  - dynamic and volatile (rapid updates, broken links)
  - immense and exponentially growing
  - hyperlinking structure (similar to citation structure)
  - real-time results (user patience remains fixed with viewing first 20 docs, while Web continues to grow $\Rightarrow$ precision must increase in pace with Web growth.)
  - lack of editorial review process (errors, lack of structure, redundancy)
  - mercantile, proprietary (advertising, financial motives)

- Web stats:
  - estimated 3 billion webpages
  - 10K each
  - doubles every 18 months
- Google stats:
  - database of 1.5 billion documents
  - receives 150 million searches/day
  - 4,000 searches/sec. during peak times
  - uses 15,000 computers
- VSM and its variants–leader among IR techniques for small, well-controlled doc collections. What is common Web IR technique?

  Direct search + post-sort by preassigned PageRank for each d_i.

Future Work

- Combine LSI (semantic structure) with MC (link structure).

- Use SANs to cluster WWW MC and save storage.

- Use MFPT from MC derived from VSM-LSI model to find nearest neighbors.

Related Fields

- Information filtering
- Image retrieval
- Multimedia/Sound retrieval
- Multilingual/cross-language retrieval

The End