Ranking with Optimization Techniques

March 21, 2010
SIAM-SEAS

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March 21, 2010
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Outline

- Minimum Violations Ranking
  - data differential matrices
  - hillside form

- Evolutionary Optimization

- BILP

- LP

- Applications
Data Differential Matrices

data on pairwise comparisons between items (e.g., teams)

Data:
• points
• rebounds
• fge
• turnovers

Points Examples:
• \(d_{ij} = 10\) and \(d_{ji} = 0\), if I beats j by 10
• \(d_{ij} = 10\) and \(d_{ji} = -10\), if I beats j by 10

For multiple games between teams, take average or cumulative differentials.
Hillside Form

Perfect Hillside Form:
  • each row is an increasing sequence
  • each column is an decreasing sequence
  • (optional) zeros on lower triangular
Hillside Form

Perfect Hillside Form:
• each row is an increasing sequence
• each column is an decreasing sequence
• (optional) zeros on lower triangular

Mathematically, $P_{mxn}$ is in perfect hillside form if:

$$P_{i,j} = 0 \quad \forall \ i \leq j : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$$

$$P_{i,k} \leq P_{j,k} \quad \forall \ i < j : i \in \{1, \ldots, m\}, j \in \{1, \ldots, m\}, k \in \{1, \ldots, n\}$$

$$P_{k,i} \leq P_{k,j} \quad \forall \ i < j : i \in \{1, \ldots, n\}, j \in \{1, \ldots, n\}, k \in \{1, \ldots, m\}$$

Example of perfect hillside form
Hillside Form and Ranking

In a perfect season, the 1\textsuperscript{st} place team beats the 2\textsuperscript{nd} place team by a little, the 3\textsuperscript{rd} place team by a little more, and so on.
Hillside Form and Ranking

In a perfect season, the 1\textsuperscript{st} place team beats the 2\textsuperscript{nd} place team by a little, the 3\textsuperscript{rd} place team by a little more, and so on.

GOAL: symmetrically reorder rows and columns of data matrix to bring it as close to hillside form as possible.

Reordering that does this = ranking vector
Hillside Form and Ranking

2009 SoCon BB teams

Original ordering  after Reordering
Violations to Hillside Form

for a reordering (aka, ranking vector), count the number of violations to hillside form
Violations to Hillside Form

for a reordering (aka, ranking vector), count the number of violations to hillside form

Original ordering

# violations = 1329

after Reordering

# violations = 351
Hillside Form and Ranking

In a perfect season, the 1\textsuperscript{st} place team beats the 2\textsuperscript{nd} place team by a little, the 3\textsuperscript{rd} place team by a little more, and so on.

GOAL: symmetrically reorder rows and columns of data matrix to bring it as close to hillside form as possible.

Reordering that does this = ranking vector

Optimization!
Solving the Optimization Problem

1. EO (evolutionary optimization)
2. BILP (binary integer linear program)
3. LP (linear program)
Evolutionary Optimization

Of all n! permutation vectors, find the one that minimizes the number of hillside violations.
Evolutionary Optimization

Of all $n!$ permutation vectors, find the one that minimizes the number of hillside violations.

Every ranking vector is a permutation vector.
A New Algorithm for Ranking Sports Teams Using Evolutionary Optimization

Kathryn Pedings (kathryn.pedings@gmail.com) and Dr. Amy Langville
College of Charleston Department of Mathematics

Abstract
Evolutionary optimization is an algorithm using Darwinian ideas of mating, mutating, fitness, and survival of the fittest. Its use has been limited to intractable problems, but it was our belief that the algorithm could be modified to be successful with tractable problems such as sports ranking.

1. Never before used on sports ranking.
2. We have shown that with some datasets our evolutionary algorithm is better than other well-known algorithms.
3. Sophisticated changes made to speed up the algorithm.

Hillside Form and Point-Differential Matrices

Initial Population

2 3 1 1
4 2 2 3
1 1 4 4
3 4 3 2

Calculate the Fitness

Get an Offspring

Mating

2 3 1 1
4 2 2 3
1 1 4 4
3 4 3 2

Mutating

1 4 3
2 1 3
4 1 4
Then calculate the fitness one step at a time.

Fitness is Bad

Fitness is Good

Population Changes Drastically

1 3 2 4
3 2 1 3
2 4 3 1
4 1 4 2

Fitness is Bad

Team 1 beat Team 4 by 15 points.

Population Changes Very Little

1 1 1 1
3 3 2 4
2 4 3 3
4 2 4 2

STOP! Print out the best parent solution; this is your team ranking!

Results
The dataset used is the basketball data for the Southern Conference for the 2007-2008 season.

Massey Ranking

Evolutionary Optimization

Colley Ranking

# of Viol = 269
# of Viol = 265
# of Viol = 278

Davidson
UNC-G
GA So.
App St.
Chatt.
CofC
Elon
Wofford
W Car.
Citadel

Furman

Future Work

• Use to rank other items such as books, movies, graduate student applicants, etc.
• Enter a bracket to ESPN for March Madness.

Acknowledgements
• Emmie Douglas for her input and scholarly advice.
• Dr. Amy Langville for her original Evolutionary Optimization code, datasets, and continued guidance throughout the research process.

Work Cited
Evolutionary Optimization

Pros:

1. Simple structure, easy to code.
2. Can handle large datasets.
3. With enough randomization and time, will reach global solution.
4. Early termination gives meaningful answer.

Cons:

1. Can get stuck in local solution for a while.
2. Can take a long time to converge.
3. Due to randomization, different runs produce different solutions.
Solving the Optimization Problem

1. EO (evolutionary optimization)
2. BILP (binary integer linear program)
3. LP (linear program)
Finds global minimum, solves MVR problem optimally.
BILP

Finds global minimum, solves MVR problem optimally.

Feb. 2009 trip to University of Tsukuba
BILP

Finds global minimum, solves MVR problem optimally.

YOSHI’S RANK AGGREGATION BILP

\[ x_{ij} = \begin{cases} 
1 & \text{if team i is ranked above team j} \\
0 & \text{otherwise} 
\end{cases} \]

maximize \[ \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} \]

subject to \[ x_{ij} + x_{ji} = 1 \quad \forall \text{ distinct pairs } (i, j) \in N \times N \]
\[ x_{ij} + x_{jk} + x_{ki} \leq 2 \quad \forall \text{ distinct triples } (i, j, k) \in N \times N \times N \]
\[ x_{ij} \in \{0,1\} \quad \forall (i, j) \in N \times N \]
\[ x_{ii} = 0 \quad \forall i \in N \]

maximizes agreement among \( k \) input ranked lists

\[ C_{ij} = \# \text{ of input lists having } i \text{ above } j \]
Finds global minimum, solves MVR problem optimally.

Yoshi’s Rank Aggregation BILP

\[ x_{ij} = \begin{cases} 1 & \text{if team } i \text{ is ranked above team } j \\ 0 & \text{otherwise} \end{cases} \]

maximizes agreement among \( k \) input ranked lists

\[ C_{ij} = \# \text{ of input lists having } i \text{ above } j \]
MVR and Rank Aggregation

Every team ranks its opponents with its point differentials, from both an offensive and a defensive viewpoint.

$max$ agreement $= min$ disagreement $= min$ # violations

**Definition of $C$ matrix**

\[
c_{ij} := \left| \left\{ k \in N \mid d_{kj} < d_{ki} \right\} \right| + \left| \left\{ k \in N \mid d_{ik} < d_{jk} \right\} \right|
\]

\[
w_{ij} := \sum_{k \in N, d_{kj} < d_{ki}} (d_{kj} - d_{ki}) + \sum_{k \in N, d_{ik} < d_{jk}} (d_{ik} - d_{jk})
\]
Every team ranks its opponents with its point differentials, from both an offensive and a defensive viewpoint.

Point Differential Matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
45 & 0 & 18 & 8 & 20 \\
3 & 0 & 0 & 2 & 0 \\
31 & 0 & 0 & 0 & 0 \\
45 & 0 & 27 & 38 & 0 \\
\end{pmatrix}
\]

C Matrix

\[
\begin{pmatrix}
0 & 8 & 6 & 5 & 7 \\
0 & 0 & 0 & 0 & 2 \\
0 & 6 & 0 & 2 & 4 \\
0 & 7 & 3 & 0 & 5 \\
0 & 2 & 1 & 1 & 0 \\
\end{pmatrix}
\]

\[C_{12} = (\# \text{ of entries in col. 2 less than col. 1}) + (\# \text{ of entries in row 1 less than row 2})\]

\[= 4 + 4 = 8\]
BILP: multiple optimal solutions

Criteria for finding some m.o.s.:

1. \( C_{ij} = C_{ji} \)
2. Teams \( i \) and \( j \) are neighbors in ranked list.

Test:

Systematically descend down the ranked list searching for sets of teams satisfying criteria.
$$\text{BILP: multiple optimal solutions}$$

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2009 SoCon Example

2 two-way ties
BILP

**Pros:**

1. Optimal solution!
2. Can find some multiple optimal solutions.
3. Fast, with good initial feasible solution and bounds.

**Cons:**

1. $O(N^3)$ transitivity constraints limit runtime and problem size.
BILP

Pros:
1. Optimal solution!
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3. Fast, with good initial feasible solution and bounds.

Cons:
1. $O(N^3)$ transitivity constraints limit runtime and problem size.

Constraint relaxation is huge help here.
BILP

\[
\begin{align*}
\text{maximize} & \quad \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} \\
\text{subject to} & \quad x_{ij} + x_{ji} = 1 \quad \forall \text{ distinct pairs } (i, j) \in N \times N \\
& \quad x_{ij} + x_{jk} + x_{ki} \leq 2 \quad \forall \text{ distinct triples } (i, j, k) \in N \times N \times N \\
& \quad x_{ij} \in \{0,1\} \quad \forall (i, j) \in N \times N \\
& \quad x_{ii} = 0 \quad \forall i \in N 
\end{align*}
\]

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Constraint relaxation is huge help here.
Solving the Optimization Problem

1. EO (evolutionary optimization)
2. BILP (binary integer linear program)
3. LP (linear program)
LP

Rank Aggregation/MVR LP

minimize \[ \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} \] \( (c_{ij} \text{ can be replaced with } w_{ij}) \)

subject to
\[ x_{ij} + x_{ji} = 1 \quad \forall \text{ distinct pairs } (i, j) \in N \times N \]
\[ x_{ij} + x_{jk} + x_{ki} \leq 2 \quad \forall \text{ distinct triples } (i, j, k) \in N \times N \times N \]
\[ 0 \leq x_{ij} \leq 1 \quad \forall (i, j) \in N \times N \]
\[ x_{ii} = 0 \quad \forall i \in N \]
LP: multiple optimal solutions

2009 SoCon Example \( X \) matrix

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2 two-way ties

Fractional Locations: (3,4) and (4,3)

(1,11) and (11,1)
LP: multiple optimal solutions

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2 2-way ties
LP: multiple optimal solutions

Theorem: If fractional pairs in LP solution are *isolated*, then

1. Locations of fractional values give information on multiple optimal solutions.
2. Objective values for BILP and LP are equal.
3. If LP solution is binary, MVR solution is unique.
### LP: sensitivity analysis

#### Range on $C_{ij}$ values

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**Tight bounds:**
- 1 and 9
- 2 and 9
- 9 and 12
**LP: sensitivity analysis**

Range on $C_{ij}$ values

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Tight bounds:
1 and 9  
2 and 9  
9 and 12
LP

Pros:
1. Nearly always finds optimal solution. Otherwise, finds excellent near-optimal soln.
2. Easy identification of multiple optimal solns.
3. Fast, with good initial feasible solution and bounds.
4. Sensitivity analysis gives confidence measures.

Cons:
1. O(N³) transitivity constraints limit runtime and problem size.
Applications

Sports

1. March Madness
2. Go
3. Sumo wrestling

Recommendation systems

1. Netflix
2. Amazon

Webpages

1. Meta-search
2. Google, Yahoo, Ask
### Applications: March Madness

Table 3: Computational Results for Iterative LP method with bounding on 347 NCAA teams

<table>
<thead>
<tr>
<th>iteration</th>
<th>LP time</th>
<th>Obj. value</th>
<th>best rank</th>
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# Applications: March Madness

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Applications: March Madness

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Future Work

- Much more sensitivity analysis
- Duality from LP

Conclusions

- Ranking and RankAgg with ties and sensitivity measures
- Gains made by considering 3 different optimization viewpoints
- LOR vs. TSP