Locally Constrained Shortest Paths and an Application in Mission Planning

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ABSTRACT

We consider a generalization of the constrained shortest path problem, in which we forbid certain pairs of edges to appear consecutively on a feasible path. This problem has applications in robotics and optimal mission planning. We propose a dynamical programming heuristic for this problem and show computational results of an application in routing an aircraft through hostile space.

1. PROBLEM AND MOTIVATION

In robotics and mission planning, finding a shortest route (w.r.t. to a not necessarily euclidean distance function) on a plane region, when modeled as a graph optimization problem, does not only reduce to the easily solvable shortest path problem. Additional constraints on the curvature and the total length of the path make the problem very hard. In this paper we investigate the problem of finding shortest paths in graphs with the latter two additional constraints. We also show how to model an application in optimal mission planning as such a constrained shortest path problem and present computational results.

2. BACKGROUND AND RELATED WORK

The *constrained shortest path (CSP) problem* in a (directed or undirected) graph $G = (V, E)$ with a cost function c and a distance function d on the edges of G is to find a minimum cost path (w.r.t. to c) from a given source $s \in V$ to one or all other vertices in V, which at the same time has a total distance (w.r.t. to d) of at most some specified limit D. We assume the functions c and d both to be nonnegative. Unlike the minimum cost path problem from one source to all destinations, the CSP problem is NP-hard [4, p.214]. It has many applications in robotics [8], mission planning [9] and Quality of Service routing [5]. We refer to [3] for a survey of known algorithms for the CSP problem.

Another generalization of the shortest path problem that arises from applications, e.g. in robotics, is that of finding a curvature-constrained shortest path [1]. If the underlying structure is an embedded planar graph, then the problem is to find a shortest path between two vertices such that the angle between any two consecutive edges on the path is at least a specified minimal value α.

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3. APPROACH AND UNIQUENESS

In this paper we consider for the first time the combination of the above two problems, i.e. finding a shortest path with a specified total distance limit and with a set of pairs of edges, which are forbidden to appear consecutively on the path. We can define the *locally constrained shortest path (LCSP) problem* as follows. Given a graph $G = (V, E)$, a cost function $c : E \rightarrow \mathbb{R}^+$, a distance function $d : E \rightarrow \mathbb{R}^+$ and a distance limit $D \in \mathbb{R}^+$, a source vertex $s \in V$ and a set of forbidden pairs of edges $F \subseteq \binom{E}{2}$, find a shortest path from s to v for any $v \in V$, such that the total distance on the path is at most D and such that if e and f appear consecutively on the path, then $\{e, f\} \notin F$.

In Section 4 we present a dynamic programming heuristic for this problem as well as some computational results for an application in mission planning.

4. RESULTS AND CONTRIBUTIONS

A dynamic programming heuristic

Dijkstra's classical shortest path algorithm makes use of the fact that any shortest path between two vertices in a graph contains a shortest path between any two vertices on that path. Joksch [7] used this idea to describe an algorithm for the CSP problem, which solves the problem exactly if all the distances d_e are integers.

Our dynamic programming heuristic is based on Joksch's algorithm and works as follows. For any $v \in V$ and any $0 \leq i \leq D$, let $w[v, i]$ denote the minimum cost of a feasible (i.e. no consecutive edges are in F) path from s to v, whose distance is at most i. We also denote by $l[v, i]$ the last edge entering v on such a path. Initially, we have $w[s, i] = 0$ and $w[v, i] = \infty$ for all $0 \leq i \leq D$ and all $v \neq s$. The cost of a shortest path from s to any $v \in V$ will then be $w[v, D]$ and a shortest path can be reconstructed from the edges stored in the $l[v, i]$ variables. We compute the values $w[v, i]$ using the following recursion

$$w[v, i] = \min\{w[v, i-1], w[u, i - d_{uv}] + c_{uv} : uv \in E, \\ d_{uv} \leq i, \{uv, l[u, i - d_{uv}]\} \notin F\}.$$

The running time of the algorithm is $O((m + n \log n)D)$ (as in [7]) and the scale and rounding algorithms used for the general case (d_e not integers) of the CSP problem [5, 6] can also be applied to our heuristic.

The deficiency of our heuristic is that it does not always produce an optimal solution, even if all the distances are integers. This is because for this problem Dijkstra's necessary criterion for a shortest path does not apply, i.e. a shortest path from s to v (satisfying the additional constraints) may not contain a shortest path (w.r.t. to these constraints) from s to all vertices on that path.

In future work we would like to investigate the quality of our heuristic in applications and other approaches to solve the LCSP problem.

An application in mission planning

In this section we consider the problem of routing an aircraft through hostile air space. We consider a polygonally bounded region $\mathcal{R} \subset \mathbb{R}^2$. Inside $\mathcal{R}$ we have a finite number of circular radars and polygonally bounded no-fly zones. Given a target t inside the region $\mathcal{R}$ and a distance limit D, the objective is to find a path through the region of total length at most D, which does not go through any no-fly zone, obeys a given minimum turn radius of the aircraft and minimizes the total time that the aircraft flies through the radars.

Routing an aircraft through enemy radars has been considered before by Zabarankin et al. [9]. However, they do not consider the constraints of flying over a target and obeying a minimum turn radius. We refer to [2] for a detailed description of the problem and how it can be modeled as a LCSP problem on a grid graph in the plane. If G is the graph and c and d are the cost and distance functions in our model, we solve the LCSP problem with the source vertex being the target and the distance limit being $D/2$.

In particular, we find a path from t to every boundary point of the region that obeys the minimum turn radius and has total distance at most $D/2$. Those paths from two different boundary points which enter the target through a feasible pair of edges, can be combined to construct a feasible path from the boundary to the target and back to the boundary, which has total distance at most D. We find the minimum cost of such a feasible combined path.

Figure 1 illustrates some of our computational results. Note that the cost of the paths goes down if we increase the distance limit D on a particular instance.

If the combination of two shortest paths to the boundary violates the constraint on the minimum turn radius at the target t, we might still obtain a cheaper feasible path by adding a small detour around the target. The detour together with the two paths to the boundary is then a feasible path. An example of this is illustrated in the lower right plot in Figure 1.

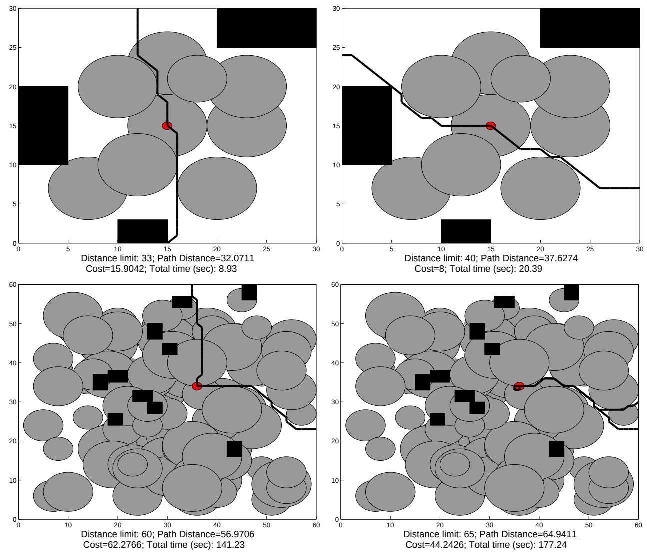

Figure 1: Outputs for the mission planning problem

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6. REFERENCES

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